$$I_{wn} = \sqrt{2D}\,\tau\,\eta \qquad D = \text{volts}^2/s$$

$$\eta \sim G(0, \sigma_\eta^2)$$

$$x = \frac{\sqrt{2D}\,\tau}{\tau} \int_{-\infty}^{t} dt' e^{-(t-t')/\tau} \eta(t')$$

$$\langle x \rangle = \langle \eta \rangle = 0$$

$$\langle x^2(t) \rangle = 2D \int_{-\infty}^{t} dt' e^{-(t-t')/\tau} \int_{-\infty}^{t} dt'' e^{-(t-t'')/\tau}$$

$$\langle \eta(t)\eta(t') \rangle$$

$$= 2D\,\sigma_\eta^2 \Delta t \int_{-\infty}^{t} dt'' e^{-2(t-t'')/\tau}$$

$$= D\tau\,\sigma_\eta^2 \Delta t$$

$$\sigma_n^2 = \frac{1}{\Delta t}$$

$$\langle \eta(t') \eta(t'') \rangle = \delta(t' - t'')$$

$$\sigma_x^2 = D\tau \qquad D = \frac{\sigma_x^2}{\tau}$$

Suppose that, on a given trial, the potential moves from V to $V + \Delta V$ in time Δt. On average, the time it takes to get to the threshold from $V + \Delta V$ must be Δt less than the time it takes from V, so

$$\langle T(V + \Delta V) \rangle = T(V) - \Delta t\,. \tag{1.34}$$

Expanding in a Taylor series,

$$\langle T(V + \Delta V) \rangle \approx T(V) + T'(V)\langle \Delta V \rangle + \frac{1}{2}T''(V)\langle \Delta V^2 \rangle\,, \tag{1.35}$$

where the primes denote derivatives with respect to V. Using equation 1.33,

$$\langle \Delta V \rangle = \frac{(V_{\text{ss}} - V)\Delta t}{\tau_{\text{m}}} \qquad \text{and} \qquad \langle \Delta V^2 \rangle = 2D\Delta t\,, \tag{1.36}$$

we find, from 1.34, that

$$DT''(V) + \frac{V_{\text{ss}} - V}{\tau_{\text{m}}}T'(V) + 1 = 0\,. \tag{1.37}$$

Defining

$$f(V) = \int^{V} dx \frac{(V_{\text{ss}} - V)}{\tau_{\text{m}} D} = -\frac{(V_{\text{ss}} - V)^2}{2\tau_{\text{m}} D} \tag{1.38}$$

so that $f'(V) = (V_{\text{ss}} - V)/\tau_{\text{m}} D$, we can write down the solution to this equation as

$$T'(V) = -\frac{e^{-f(V)}}{D}\int_{-\infty}^{V} dy\, e^{f(y)}\,. \tag{1.39}$$

Integrating this result, we find

$$T(V) = -\frac{1}{D}\int_{V_{\text{th}}}^{V} dx\, e^{-f(x)} \int_{-\infty}^{x} dy\, e^{f(y)}\,, \tag{1.40}$$

where we have imposed the firing condition $T(V_{\rm th}) = 0$. This means that the answer we seek is

$$\frac{1}{R} = \frac{1}{D}\int_{V_{\rm reset}}^{V_{\rm th}} dx\, e^{-f(x)} \int_{-\infty}^{x} dy\, e^{f(y)}\,. \tag{1.41}$$

With some substitution, this can be written as

$$\frac{1}{R} = \frac{\tau_{\rm m}}{\sigma_V^2}\int_{V_{\rm reset}}^{V_{\rm th}} dx\ \exp((V_{\rm ss}-x)^2/2\sigma_V^2) \int_{-\infty}^{x} dy\ \exp(-(V_{\rm ss}-y)^2/2\sigma_V^2)\,. \tag{1.42}$$

Finally, changing variables $y \to (y - V_{\rm ss})/\sqrt{2}\sigma_V$ and $x \to (x - V_{\rm ss})/\sqrt{2}\sigma_V$, we find

$$\frac{1}{R} = 2\tau_{\rm m}\int_{(V_{\rm reset}-V_{\rm ss})/\sqrt{2}\sigma_V}^{(V_{\rm th}-V_{\rm ss})/\sqrt{2}\sigma_V} dx\ \exp(x^2) \int_{-\infty}^{x} dy\ \exp(-y^2)\,. \tag{1.43}$$

Using the fact that

$$\int_{-\infty}^{x} dy\ \exp(-y^2) = \frac{\sqrt{\pi}\,(1+\mathrm{erf}(x))}{2}\,, \tag{1.44}$$

we obtain the final result

$$\frac{1}{R} = \tau_{\rm m}\sqrt{\pi}\int_{(V_{\rm reset}-V_{\rm ss})/\sqrt{2}\sigma_V}^{(V_{\rm th}-V_{\rm ss})/\sqrt{2}\sigma_V} dx\ \exp(x^2)\,(1+\mathrm{erf}(x))\ . \tag{1.45}$$

Useful Numerical Approximation

The integral in equation 1.45 is difficult to compute numerically because of the nature of the integrand $\exp(x^2)(1+\mathrm{erf}(x))$. To compute this integral using standard methods, use the following approximation.

$$\exp(x^2)(1+\mathrm{erf}(x)) \approx \begin{cases} f_1 & \text{if } x \le 0 \\ 2\exp(x^2) - f_1 & \text{if } x > 0\,, \end{cases} \tag{1.46}$$

where

$$f_1 = t\exp(\alpha)\,, \qquad t = \frac{1}{1+0.5|x|}\,, \tag{1.47}$$

and

$$\alpha = a_1 + t(a_2 + t(a_3 + t(a_4 + t(a_5 + t(a_6 + t(a_7 + t(a_8 + t(a_9 + t a_{10})))))))) \tag{1.48}$$

with

$$\begin{aligned} a_1 &= -1.26551223 \quad a_2 = 1.00002368 \quad a_3 = 0.37409196 \\ a_4 &= 0.09678418 \quad a_5 = -0.18628806 \quad a_6 = 0.27886087 \\ a_7 &= -1.13520398 \quad a_8 = 1.48851587 \quad a_9 = -0.82215223 \\ & \qquad a_{10} = 0.17087277 \end{aligned} \tag{1.49}$$